

THE MODAL LOGIC OF σ -CENTERED FORCING AND RELATED FORCING CLASSES

UR BEN-ARI-TISHLER

ABSTRACT. We consider the modality “ φ is true in every σ -centered forcing extension”, denoted $\Box\varphi$, and its dual “ φ is true in some σ -centered forcing extension”, denoted $\Diamond\varphi$ (where φ is a statement in set theory), which give rise to the notion of a *principle of σ -centered forcing*. We prove that if ZFC is consistent, then the modal logic of σ -centered forcing, i.e. the ZFC-provable principles of σ -centered forcing, is exactly S4.2. We also generalize this result to other related classes of forcing.

1. INTRODUCTION

Modal logic is used to formalize various modalities appearing in language, the most common of which are the notions of possibility and necessity (alethic modalities), but also temporal modalities such as “always”, doxastic modalities such as “believes” and many more. Given a certain formal language for modal logic, one can adjoin an intended interpretation to the modal operators, and with this intended interpretation in mind, find the right modal theory to fit one’s needs. Besides these meta-theoretical semantics, modal logic has a few formal semantics, the most prominent of which is the “possible worlds semantics”, or Kripke semantics, which give formal meaning to the intuition that something is possible if “it is true in some possible alternative world”, and necessary if “it is true in all possible worlds”. These semantics are based on the so-called Kripke models, which consist of a set of “worlds”, related by an “accessibility relation” where a statement is possible in some world if there is a world accessible to it where the statement is true; and similarly, a statement is necessary if it is true in all accessible worlds. These semantics are highly reminiscent of the situation in set-theoretic forcing, where we have models of set-theory, related to one another by the relation of “being a forcing extension”. This suggests a forcing interpretation of modal logic, where a statement is said to be possible, or forceable, if it is true in some forcing extension (of the universe), and necessary if it is true in all forcing extensions. This interpretation was first suggested by Joel Hamkins in [4], where he used it to formalize a new forcing axiom called the “Maximality Principle” (MP). With this interpretation in place, the question arose - what modal theory best captures this interpretation? In other words - what is the “Modal Logic of Forcing”? This question actually splits into two - first, what modal principles of forcing are provable in ZFC? And second, given a specific model of ZFC, what are the modal principles true in it? These questions were formally asked by Hamkins and Benedikt Löwe in [6], where they answered the first question by showing that the ZFC-provable principles of forcing are exactly the modal theory known as S4.2 (see below). They also began addressing the

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second question, by showing, e.g., that the modal principles true in any specific model contain S4.2 and are contained in S5, and that both ends of the spectrum are realized (by L on one hand and by Hamkins's model for the Maximality Principle on the other). Another related question is - what happens if we consider only a specific class of forcing notions, such as the class of all c.c.c forcings? This question was addressed more extensively in [5], where it was shown, e.g., that the ZFC-provable principles of collapse forcing are exactly S4.3. In this work we continue this line of inquiry, with respect to the class of all σ -centered forcing notions. Thus, the main result of this work is that the modal logic of σ -centered forcing is exactly the modal theory S4.2. After establishing this main theorem, we show that the techniques developed here can be adjusted to other related classes of forcing notions.

We begin by setting some preliminaries - first we cite common definitions and theorems of forcing and of modal logic; and then present the main tools developed in [6, 5] for the research of the modal logic of forcing; we add one new notion to this set of tools, the notion of an n -switch, and show it's utility. Then we present the class of σ -centered forcing, prove some of its properties which give us the easy part of the theorem - that the modal logic of σ -centered forcing contains S4.2, and present the technique of coding subsets using σ -centered forcing. The hard part of the main theorem will be proved in section 4, where we begin by defining a specific model of ZFC, and then present two forcing constructions that would allow us to establish that the modal logic of σ -centered forcing is contained in S4.2. We conclude with the above-mentioned generalizations and some open questions.

2. PRELIMINARIES

We begin by presenting some notations and background that will be used in this work.

2.1. Forcing. Our forcing notation will usually follow Kunen's [10, chapter VII]. A poset $\mathbb{P} = (P, \leq)$ will usually be confused with it's underlying set. We say that p is *stronger* than q , or p *extends* q , if $p \leq q$. Unless stated otherwise, every poset is assumed to have a unique maximal element, denoted $1_{\mathbb{P}}$ or just 1 if the context is clear. We denote compatible elements by $p \parallel q$ and incompatible by $p \perp q$. We usually denote \mathbb{P} -names by letters τ, σ etc., or with a dot - \dot{x} . The canonical names for elements of the ground model are denoted by "check" - \check{x} . For a model M , we will sometimes denote by $M^{\mathbb{P}}$ just any generic extension of M by a \mathbb{P} -generic filter.

In this work, Cohen forcing \mathbb{C} is the poset $\omega^{<\omega}$, where $s \leq t$ iff t is an initial segment of s (denoted $t \leq s$). We denote the length of $s \in \omega^{<\omega}$ by $\ell(s)$. If $G \subseteq \mathbb{C}$ is a generic filter, then $f = \bigcup G$ is in ω^ω and is called a Cohen generic real. Any real $r \in \omega^\omega$ is called Cohen generic if there is some generic $H \subseteq \mathbb{C}$ such that $r = \bigcup H$. This means that r is generic if every dense open set in $\omega^{<\omega}$ meets some finite initial segment of r .

We can extend the notion of "dense-open set" to ω^ω :

- Definition 2.1.** (1) A *basic open set* (in ω^ω) is a set of the form $U_s := \{x \in \omega^\omega \mid s \leq x\}$ for some $s \in \omega^{<\omega}$;
 (2) A set $U \subseteq \omega^\omega$ is *open* if for every $x \in U$ there is some $s \in \omega^{<\omega}$ such that $x \in U_s \subseteq U$;
 (3) A set $D \subseteq \omega^\omega$ is *dense* if it meets every basic-open set.

A real r is Cohen generic over a model M if it is in any open dense subset of ω^ω in M .

2.2. Modal logic. We provide a short account of modal logic based on [2]. The reader who is familiar with modal logic may wish to skip to definition 2.2. We work with propositional modal logic, in which formulas are constructed from propositional variables $p, q, r \dots$ or p_0, p_1, \dots ; the logical connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$; and two unary operators \Box, \Diamond . For a formula φ , $\Box\varphi$ is read “box φ ” or “necessarily φ ” and $\Diamond\varphi$ is read “diamond φ ” or “possibly φ ”. Unlike classical propositional logic, in modal logic there are many possible axiomatic systems resulting in different modal theories. A *modal theory* is a set of modal formulas containing all classical tautologies and closed under the deduction rules *modus ponens* (from φ and $\varphi \rightarrow \psi$ deduce ψ) and *necessitation* (from φ , deduce $\Box\varphi$), and under uniform substitution. We will focus on the modal theory S4.2 obtained from the following axioms:

$$\begin{array}{ll} \text{K} & \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi) \\ \text{Dual} & \Diamond\varphi \leftrightarrow \neg\Box\neg\varphi \\ \text{T} & \Box\varphi \rightarrow \varphi \\ 4 & \Box\varphi \rightarrow \Box\Box\varphi \\ .2 & \Diamond\Box\varphi \rightarrow \Box\Diamond\varphi \end{array}$$

by closing under the deduction rules.

Our semantics for modal logic are based on the notion of a *Kripke model* - a triplet $\mathcal{M} = \langle W, R, V \rangle$ where W is a non-empty set, R is a binary relation on W and V a function from the propositional variables to subsets of W . Elements of W are usually called *worlds*, and R is called *the accessibility relation*. Thus wRu is referred to as “ u is accessible to w ”. $\mathcal{F} = \langle W, R \rangle$ is called the *frame* on which \mathcal{M} is based. V , the *valuation*, assigns every propositional variable p a set of worlds $V(p) \subseteq W$, which are considered as the worlds where p is true. To make this notion precise, we define the relation $\mathcal{M}, w \models \varphi$ (“ φ is satisfied in w ”), where $w \in W$, by induction on the construction of the formula, where the atomic case is given by V , the logical connectives are defined in the obvious way, and for the modal operators we use:

- $\mathcal{M}, w \models \Box\varphi$ iff for every $u \in W$ such that wRu , $\mathcal{M}, u \models \varphi$;
- $\mathcal{M}, w \models \Diamond\varphi$ iff there exists $u \in W$ such that wRu and $\mathcal{M}, u \models \varphi$.

We say that φ is *valid in \mathcal{M}* , $\mathcal{M} \models \varphi$, if $\mathcal{M}, w \models \varphi$ for every $w \in W$, and that φ is *valid on a frame \mathcal{F}* , $\mathcal{F} \models \varphi$, if φ is valid in every model based on \mathcal{F} .

So far, no assumptions on the accessibility relation were made. And in fact, there is a strong connection between properties of the relation and the valid formulas of the model. Let Λ be a modal theory, and \mathcal{C} some class of frames. We say Λ is *sound with respect to \mathcal{C}* if every $\varphi \in \Lambda$ is valid in every frame in \mathcal{C} . Λ is *complete with respect to \mathcal{C}* if every formula valid in every frame of \mathcal{C} is in Λ ; equivalently, for every $\varphi \notin \Lambda$ there is some model $\mathcal{M} = \langle W, R, V \rangle$ where $\langle W, R \rangle \in \mathcal{C}$ and some $w \in W$ such that $\mathcal{M}, w \models \neg\varphi$. Λ is *characterized by \mathcal{C}* if it is sound and complete with respect to it.

A well known theorem is that the modal theory we are interested in, S4.2, is characterized by the class of finite frames $\langle W, R \rangle$ for which R is reflexive, transitive and directed (i.e. $wRv \wedge wRu \rightarrow \exists z(vRz \wedge uRz)$). However, it will be convenient to find a class of more structured frames characterizing this theory.

Definition 2.2. Let $\langle F, \leq \rangle$ such that \leq is a reflexive and transitive binary relation on F . We can define an equivalence relation on F by setting $x \equiv y$ iff $x \leq y \leq x$. This equivalence relation induces a partial order on F/\equiv , which we will also denote by \leq . $\langle F, \leq \rangle$ is called a *pre-Boolean-algebra* (a pBA) if $\langle F/\equiv, \leq \rangle$ is a Boolean-algebra (a BA). Note that a pBA is in particular reflexive, transitive and directed.

A pBA can be thought of as a BA where every element is replaced by a cluster of equivalent elements.

Theorem 2.3 (Theorem 11 in [6]). *S4.2 is characterized by the class of all finite pre-Boolean-algebras.*

2.3. The modal logic of forcing. We present the framework of the modal logic of forcing, based on [6] and [5]. The reader who is familiar with these works may wish to skip to definition 2.12 where we define the new notion of an n -switch.

As we mentioned in the introduction, the possible world semantics suggest a connection between modal logic and forcing, as we can imagine all generic extensions of the universe (or of a specific model of ZFC) as an enormous Kripke model (called “*the generic multiverse*”). This leads to the forcing interpretation of modal logic, in which we say that a sentence of set-theory φ is necessary ($\Box\varphi$) if it is true in all forcing extensions, and possible ($\Diamond\varphi$) if it is true in some forcing extension. Note that these operators can be expressed in the language of set-theory by the definability of the forcing relation: $\Box\varphi$ is the statement “for every poset \mathbb{P} and $p \in \mathbb{P}$, $p \Vdash \varphi$ ” and $\Diamond\varphi$ is the statement “there is a poset \mathbb{P} and $p \in \mathbb{P}$ such that $p \Vdash \varphi$ ”. Given some definable class of forcing notions Γ , we can also restrict to posets belonging to that class, to get the operators $\Box_\Gamma, \Diamond_\Gamma$. The following definitions, based on [6] and [5], allow us to formally ask the question - what statements are valid under this interpretation?

Definition 2.4. (1) Given a formula $\varphi = \varphi(q_0, \dots, q_n)$ in the language of modal logic, where q_0, \dots, q_n are the only propositional variable appearing in φ , and some set-theoretic sentences ψ_0, \dots, ψ_n , the *substitution instance* $\varphi(\psi_0, \dots, \psi_n)$ is the set-theoretic statement obtained recursively by replacing q_i with ψ_i and interpreting the modal operators according to the forcing interpretation (or the Γ -forcing interpretation).

(2) Let Γ be a class of forcing notions. The *ZFC-provable principles of Γ -forcing* are all the modal formulas φ such that $\text{ZFC} \vdash \varphi(\psi_0, \dots, \psi_n)$ for every substitution $q_i \mapsto \psi_i$ under the Γ -forcing interpretation. This will also be called *the modal logic of Γ -forcing*, denoted $\text{MLF}(\Gamma)$. If we discuss the class of all forcing notions we omit mention of Γ .

Theorem 2.5 (Hamkins and Löwe, [6]). *If ZFC is consistent then the ZFC-provable principles of forcing are exactly S4.2.*

We will now present the main tools which were developed to prove the theorem above, and which can be used to prove similar theorems. To prove such a theorem, we need to establish lower and upper bounds, i.e. find a modal theory Λ such that $\text{MLF}(\Gamma) \supseteq \Lambda$ and $\text{MLF}(\Gamma) \subseteq \Lambda$ respectively. Each type of bound require a different set of tools, which will be presented below.

2.3.1. Lower bounds. A simple observation is that the ZFC-provable principles of Γ -forcing are closed under the deduction rules - modus ponens is obvious, but also necessitation, since if every substitution instance of φ is provable, it is true in every model, and in particular in every generic extension, so also every substitution of $\Box\varphi$ is provable. So if a modal theory is given by some axioms, to show it is contained in $\text{MLF}(\Gamma)$ it is enough to check that the axioms are valid principles of Γ -forcing. So for example, axioms K and Dual are easily seen to be valid under the Γ -forcing interpretation for every class Γ . The validity of other axioms depends on specific properties of Γ .

Definition 2.6. A definable class of forcing notions Γ is said to be *reflexive* if it contains the trivial forcing; *transitive* if it is closed under finite iterations, i.e. if $\mathbb{P} \in \Gamma$ and \dot{Q} is a \mathbb{P} -name for a poset such that $\Vdash_{\mathbb{P}} \dot{Q} \in \Gamma$, then $\mathbb{P} * \dot{Q} \in \Gamma$; *persistent* if $\mathbb{P}, \mathbb{Q} \in \Gamma$ implies $\mathbb{Q} \in \Gamma^{V^{\mathbb{P}}}$; and *directed* if $\mathbb{P}, \mathbb{Q} \in \Gamma$ implies that there is some $\mathbb{R} \in \Gamma$ such that \mathbb{R} is forcing equivalent to $\mathbb{P} * \dot{S}$ and to $\mathbb{Q} * \dot{T}$, where $\dot{S} \in \Gamma^{V^{\mathbb{P}}}$ and $\dot{T} \in \Gamma^{V^{\mathbb{Q}}}$.

Note that if a Γ is transitive and persistent, we can show it is directed by taking $\mathbb{R} = \mathbb{P} \times \mathbb{Q}$ for any $\mathbb{P}, \mathbb{Q} \in \Gamma$.

Example 2.7. The class of all forcing notions is reflexive, transitive and persistent (and thus directed). The class of all c.c.c forcing notions is reflexive and transitive, but not persistent, and in fact not directed.

Theorem 2.8 (Thm. 7 in [5]). *Axiom T is valid in every reflexive forcing class, axiom 4 in every transitive forcing class and axiom .2 in every directed forcing class. Thus, if Γ is reflexive, transitive and directed then $\text{MLF}(\Gamma) \supseteq \text{S4.2}$.*

2.3.2. Upper bounds. To establish that Λ is an upper bound for $\text{MLF}(\Gamma)$, we need to show that every formula not in Λ is also not in $\text{MLF}(\Gamma)$. To do so, we would need to find a model of ZFC and some substitution instance of φ that fails in this model. In the case that Λ is complete w.r.t some class of frames \mathcal{C} , $\varphi \notin \Lambda$ means that there is some Kripke model based on a frame in \mathcal{C} where φ fails. So our goal would be to find a suitable model of set theory W such that the Γ -generic multiverse generated by W (i.e. all Γ -forcing extensions of W) “looks like” the model where φ fails. The main tool for that is called a labeling:

Definition 2.9. A Γ -labeling of a frame $\langle F, R \rangle$ for a model of set-theory W is an assignment to each $w \in F$ a set-theoretic statement Φ_w such that:

- (1) The statements form a mutually exclusive partition of truth in the Γ -generic multiverse over W , i.e. every Γ -generic extension of W satisfies exactly one Φ_w .
- (2) The statements correspond to the relation, i.e. if $W[G]$ is a Γ -forcing extension of W that satisfies Φ_w , then $W[G] \models \Diamond \Phi_u$ iff wRu .
- (3) $W \models \Phi_{w_0}$ where w_0 is a given initial element of F .

Lemma 2.10 (The labeling lemma - Lemma 9 in [5]). *Suppose $w \mapsto \Phi_w$ is a Γ -labeling of a finite frame $\langle F, R \rangle$ for a model of set-theory W with w_0 an initial world of F , and \mathcal{M} a Kripke model based on F . Then there is an assignment of the propositional variables $p \mapsto \psi_p$ such that for every modal formula $\varphi(p_0, \dots, p_n)$*

$$\mathcal{M}, w_0 \models \varphi(p_0, \dots, p_n) \quad \text{iff} \quad W \models \varphi(\psi_{p_0}, \dots, \psi_{p_n}).$$

Corollary 2.11. *If every finite pre-Boolean-algebra has a Γ -labeling over some model of ZFC, then $\text{MLF}(\Gamma) \subseteq \text{S4.2}$*

Proof. By theorem 2.3, every modal formula $\varphi \notin \text{S4.2}$ fails in a Kripke model based on some finite pBA. So, given a Γ -labeling for this frame over a model W , by the labeling lemma there is a substitution instance of φ which fails at W under the Γ -forcing interpretation. So $\varphi \notin \text{MLF}(\Gamma)$. \square

Hence to establish upper bounds, we try to find labelings for specific frames. Various labelings can be constructed using certain kinds of set-theoretic statements, called in general *control statements*.

Definition 2.12 (Control statements). Let W be some model of set theory, and Γ some class of forcing notions.

- (1) A *switch* for Γ -forcing over W is a statement s such that necessarily, both s and $\neg s$ are possible. That is, over every extension of W one can force s or $\neg s$ as one chooses.
- (2) An n -*switch* for Γ -forcing over W is a set of statements $\{s_i \mid i < n\}$ (where $n > 1$) such that every Γ -generic extension W' of W satisfies exactly one s_i , and if $W' \models s_i$ and $j \neq i$ then there is a Γ -forcing extension of W' satisfying s_j . The n -switch value in some $W[G]$ is the i such that $W[G] \models s_i$. Note that a 2-switch is just a switch.
- (3) A *button* for Γ -forcing over W is a statement b which is necessarily possibly necessary, i.e. $W \models \Box \Diamond \Box b$. This means that in every extension of W , we can force b to be true and to remain true in every further extension. A button is called *pushed* if $\Box b$ holds, otherwise it is called *unpushed*. A *pure button* is a button b such that $\Box(b \rightarrow \Box b)$ (i.e. if it is true then it is pushed). If b is an unpushed button then $\Box b$ is an unpushed pure button.
- (4) A *ratchet* for Γ -forcing over W is a collection of pure buttons $\{r_i \mid i \in I\}$, possibly with i as a parameter, where I is well-ordered, such that pushing r_i pushes every r_j for $j < i$, and necessarily, every unpushed r_i can be pushed without pushing any r_j for $j > i$. An infinite ratchet $\{r_i \mid i \in I\}$ is called *strong* if there is no Γ -extension of W satisfying every r_i . The ratchet value in $W[G]$ is the first $i \in I$ such that $W[G] \models \neg r_i$.
- (5) A family of control statements (switches, n -switches, buttons, ratchets) is called *independent over W* (for Γ -forcing) if in W , all buttons are unpushed (including the ones in any ratchet), and necessarily, using Γ -forcing, each button can be pushed, each switch can be turned on or off, the value of each n -switch can be changed, and the value of every ratchet can be increased, without affecting any other control statement in the family.

Note the “necessarily” - the independence needs to be preserved in any Γ -forcing extension of W .

n -switches are less naturally occurring in set-theory than the other notions, and indeed they were not explicitly defined in [6] and [5]. However, by examining the proofs of some of the main theorems there, one can see that what was implicitly used was an n -switch, which was constructed using switches (cf. [5, theorems 10,11,13]). Additionally, in some cases switches were constructed from ratchets and then transformed into n -switches (e.g. in [5, theorems 12, 15]). So, in the definition of some of the central labelings, n -switches turn out to be the more natural notion, and we will show how to construct them using either switches or a ratchet independently. Hence the following theorem, which gives sufficient conditions for the existence of labelings for finite pBA's, generalizes some of the above-mentioned theorems from [5], and they can be inferred from it. We will not be able to use the theorem as it is to prove our main theorem, but we will use its proof as a model, so it has instructive value in itself.

Theorem 2.13. *Let Γ be some reflexive and transitive forcing class and W a model of set theory. If there are arbitrarily large finite families of buttons, mutually independent of n -switches for large enough n 's, then there is a Γ -labeling over W for every frame which is a finite pre-Boolean-algebra.*

Proof. Let $\langle F, \leq \rangle$ be a finite pBA. As noted earlier, it can be viewed a finite BA, where each element is replaced by a cluster of equivalent worlds. We can add dummy worlds to each cluster without changing satisfaction in the model, so we can assume that each

cluster is of size n for some $1 < n < \omega$. It is known that any finite BA is isomorphic to the BA $\langle \mathcal{P}(B), \subseteq \rangle$ for some finite set B . Let B be such that $\langle F, \equiv, \leq \rangle \cong \langle \mathcal{P}(B), \subseteq \rangle$, and set $m = |B|$. We can assume that in fact $B = \{0, \dots, m-1\}$. There is a correspondence between subsets $A \subseteq B$ and clusters in $\langle F, \leq \rangle$. Each cluster is of size n , so by enumerating each cluster, all the elements of F can be named w_i^A for $i < n$ and $A \subseteq B$, where $w_i^A \leq w_j^{A'}$ iff $A \subseteq A'$. An initial world in F must be in the bottom cluster, which corresponds to $\emptyset \subseteq B$ so WLoG it is enumerated as w_0^\emptyset .

By the assumption, adding more dummy worlds to each cluster if needed, there are buttons $\{b_0, \dots, b_{m-1}\}$ and an n -switch $\{s_0, \dots, s_{n-1}\}$ all independent of each other over W . We can assume the buttons are pure. To define a labeling, each cluster, corresponding to some $A \subseteq B$, will be labeled by the statement that the only buttons pushed are the ones with indexes from A . Inside each cluster, each world will be labeled by the corresponding value of the n -switch. Formally, we set

$$\Phi(w_i^A) = \bigwedge_{j \in A} b_j \wedge \bigwedge_{j \notin A} \neg b_j \wedge s_i$$

and claim that this is a labeling as required by verifying the conditions:

- (1) If $W[G]$ is a Γ -generic extension of W , define $A = \{j < m \mid W[G] \models b_j\}$. By the definition of the n -switch, $W[G] \models s_i$ for some unique $i < n$. So it is clear that $W[G] \models \Phi(w_i^A)$, and that for any other pair $(A', i') \neq (A, i)$ with $A' \subseteq B$, and $i' < n$, $W[G] \not\models \Phi(w_{i'}^{A'})$. So these statements indeed form a mutually exclusive partition of truth in the Γ -generic multiverse over W .
- (2) Assume $W[G]$ is a Γ -generic extension of W such that $W[G] \models \Phi(w_i^A)$.
 If $w_i^A \leq u$, then as we have seen, $u = w_{i'}^{A'}$ for some $i' < n$ and $A \subseteq A' \subseteq B$. By the assumption of independence of the control statements, we can, by Γ -forcing, push all the buttons in $A' \setminus A$ (and only them) and change the n -switch value to i' (if needed), to obtain an extension of $W[G]$ satisfying $\Phi(w_{i'}^{A'})$. Note that by the transitivity of Γ , the b_i 's are still independent pure buttons in $W[G]$, since every Γ -extension of $W[G]$ is also a Γ -extension of W . In particular, any button true in $W[G]$ remains true in the extension. So $W[G] \models \Diamond \Phi(w_{i'}^{A'})$ as required.
 If $W[G] \models \Diamond \Phi(w_{i'}^{A'})$, then there is some extension $W[G][H] \models \Phi(w_{i'}^{A'})$. By the definition of pure buttons and the reflexivity of Γ , $W[G] \models \bigwedge_{j \in A} b_j$ implies $W[G] \models \bigwedge_{j \in A} \Box b_j$, so $W[G][H] \models \bigwedge_{j \in A} b_j$. Therefore by the definition of $\Phi(w_{i'}^{A'})$, we must have $A \subseteq A'$, so $w_i^A \leq w_{i'}^{A'}$.
- (3) A part of the definition of independence is that no button is pushed in W (since they are pure and Γ reflexive, it is equivalent to saying none is true). We can assume WLoG that $W \models s_0$. So $W \models \Phi(w_0^\emptyset)$. \square

Corollary 2.14. *Under the assumptions of theorem 2.13, $\text{MLF}(\Gamma) \subseteq \text{S4.2}$.*

Proof. Apply corollary 2.11. \square

Lemma 2.15. *An n -switch can be produced using the following control statements:*

- (1) *Independent switches s_0, \dots, s_{m-1} if $n = 2^m$;*
- (2) *A strong ratchet $\{r_i \mid i \in I\}$ where I is either a limit ordinal or Ord , the class of all ordinals, and $i \in I$ is a parameter in r_i .*

Proof. For (1), if $j < 2^m$ let \bar{s}_j be the statement that the pattern of switches corresponds to the binary digits of j , that is,

$$\bigwedge \{s_i \mid \text{the } i\text{-th binary digit of } j \text{ is } 1\} \wedge \bigwedge \{\neg s_i \mid \text{the } i\text{-th binary digit of } j \text{ is } 0\}.$$

Clearly in any extension exactly one pattern of the switches holds, so exactly one \bar{s}_j holds. By the independence of the switches, any pattern can be forced over any extension.

For (2), every $i \in I$ is an ordinal, so of the form $\omega \cdot \alpha + k$ for some $\alpha \in \text{Ord}$ and $k < \omega$. Then we let \bar{s}_j be the statement “if $i = \omega \cdot \alpha + k$ is the first such that $\neg r_i$ then $k \bmod n = j$ ”. Since no extension satisfies all the r_i ’s, there is always some i which is the first such that $\neg r_i$, and therefore there is some unique j such that \bar{s}_j holds. Since it is a ratchet, in every extension, for every $j' < n$, we can increase its value to some $i' = \omega \cdot \alpha' + k'$ for some $k' > k$ such that $k' \bmod n = j$ (we use the assumption that if I is an ordinal then it is a limit). \square

Note also that every family of independent buttons $\langle b_i \mid i \in I \rangle$ where I is as above, with no extensions where all of them are pushed, can be transformed into a strong ratchet by setting $r_i = \forall j < i \, b_j \wedge \neg b_i$. To conclude, we get the following:

Corollary 2.16 ([5, theorems 13 and 15]¹). *Let Γ be some reflexive and transitive forcing class and W a model of set theory. If there are arbitrarily large finite families of buttons mutually independent with arbitrarily large finite families of switches, with a strong ratchet as above or with another family of independent buttons as above, then there is a Γ -labeling for every frame which is a finite pre-Boolean-algebra over W . So in such cases, $\text{MLF}(\Gamma) \subseteq \text{S4.2}$.*

3. σ -CENTERED FORCING

We now proceed to the investigation of the modal logic of a specific class of forcing notions - the class of all σ -centered forcing notions.

Definition 3.1. Let \mathbb{P} be any poset.

- (1) A subset $C \subseteq \mathbb{P}$ is called *centered* if any finite number of elements in C have a common extension in \mathbb{P} .
- (2) A poset is called *σ -centered* if it is the union of countably many centered subsets.

Remark 3.2. For convenience we will always assume that the top element $1_{\mathbb{P}}$ is in each of the centered posets. This does not affect the generality since every element is compatible with it. It will also sometimes be convenient to assume that if $\mathbb{P} = \bigcup_{n \in \omega} \mathbb{P}_n$ where each \mathbb{P}_n is centered, then each \mathbb{P}_n is upward closed, i.e. if $q \in \mathbb{P}_n$ and $q \leq p$ then $p \in \mathbb{P}_n$. This also doesn’t affect the generality since if $q_1, \dots, q_k \in \mathbb{P}_n$ and $q_i \leq p_i$ then a common extension for the q_i ’s will also extend the p_i ’s.

The following is a central example for a σ -centered forcing, versions of which will be used later on:

Definition 3.3. Let Y be a subset of $\mathcal{P}(\omega)$. We define a poset \mathbb{P}_Y as follows:

- The elements are of the form $\langle s, t \rangle$ where s is a finite subset of ω and t a finite subset of Y ;
- $\langle s, t \rangle$ is extended by $\langle s', t' \rangle$ if $s \subseteq s'$, $t \subseteq t'$ and for every $A \in t$, $s \cap A = s' \cap A$.

¹In [5, theorem 15] they have a slightly different convention, where the ratchet value is the last button which is pushed, and they use the notion of a *uniform ratchet*, but the theorem is essentially the same.

So we think of the first component as finite approximations for a generic real $x \subseteq \omega$, while the second component limits our options in extending the approximation. A condition $p = \langle s, t \rangle$ tells us that $s \subseteq x$ and that for every $A \in t$, $x \cap A = s \cap A$, so that the intersection of x with any set in Y will turn out to be finite.

Lemma 3.4. *For any $Y \subseteq \mathcal{P}(\omega)$, \mathbb{P}_Y is σ -centered.*

Proof. Note that if t_1, \dots, t_n are finite subsets of Y , then for any $s \in [\omega]^{<\omega}$, the conditions $\langle s, t_1 \rangle, \dots, \langle s, t_n \rangle$ are all extended by $\langle s, t_1 \cup \dots \cup t_n \rangle$. So \mathbb{P} is the union of the centered posets $\mathbb{P}_s = \{\langle s, t \rangle \mid t \subseteq Y \text{ finite}\}$. Since there are only countably many finite subsets of ω , we get that \mathbb{P} is σ -centered. \square

We will explore the properties of this kind of posets in section 3.2.

3.1. Properties of σ -centered forcing. We proceed with a few general properties of forcing with σ -centered posets. Most of them are folklore, but we provide the proofs for the benefit of the reader. An immediate observation is that every σ -centered poset also has the c.c.c, since any uncountable set will have an uncountable intersection with one of the centered posets, and therefore cannot be an antichain. So forcing with a σ -centered posets preserves all cardinals and cofinalities.

But σ -centered forcing notions have the stronger property that they also preserve the continuum function:

Lemma 3.5. *Assume $\lambda \geq \aleph_0$, $2^\lambda = \kappa$ and let \mathbb{P} be some σ -centered forcing notion. Then $V^\mathbb{P} \models 2^\lambda = \kappa$.²*

Proof. Assume $\mathbb{P} = \bigcup_{n < \omega} \mathbb{P}_n$ where each \mathbb{P}_n is centered. Let $f \in V^\mathbb{P}$ be some function from λ to 2, let \dot{f} be a name denoting f and $q \in \mathbb{P}$ s.t. $q \Vdash \dot{f} : \check{\lambda} \rightarrow \check{2}$. In V , define

$$A_f = \left\{ (\alpha, n, i) \mid \alpha < \lambda, i < 2, \exists p \in \mathbb{P}_n \left(p \leq q \wedge p \Vdash \dot{f}(\check{\alpha}) = \check{i} \right) \right\}.$$

So, if $f, g : \lambda \rightarrow 2$ are distinct, then $A_f \neq A_g$. To see this, say $f(\alpha) = 0$ and $g(\alpha) = 1$, and let p force both these facts. So, if $p \in \mathbb{P}_n$, we clearly get that $(\alpha, n, 0) \in A_f$. But also $(\alpha, n, 0) \notin A_g$, since that would imply there is some $p' \in \mathbb{P}_n$ such that $p' \Vdash \dot{g}(\check{\alpha}) = 0$, but p is compatible with p' and $p \Vdash \dot{g}(\check{\alpha}) = 1$ by contradiction. Hence 2^λ in $V^\mathbb{P}$ is bounded by the possible number of sets of the form A_f in V , which is bounded by

$$|\{(\alpha, n, i) \mid \alpha < \lambda, n < \omega, i < 2\}| \leq |\lambda \times \omega \times 2| = 2^\lambda = \kappa$$

so $V^\mathbb{P} \models 2^\lambda \leq \kappa$ (note that as we mentioned earlier, by c.c.c $\kappa^V = \kappa^{V^\mathbb{P}}$). Since every subset of λ in V is also one in $V^\mathbb{P}$, we get equality. \square

Another important property of σ -centered forcing is its productivity. Unlike c.c.c posets which are not necessarily productive, σ -centered posets are productive in a rather strong manner:

Lemma 3.6. *Let $\langle \mathbb{P}_\alpha \mid \alpha < \lambda \rangle$ for some $\lambda < (2^{\aleph_0})^+$ be a collection of σ -centered posets. Let $\mathbb{P} = \prod_{\alpha < \lambda} \mathbb{P}_\alpha$ be the finite support product of $\langle \mathbb{P}_\alpha \mid \alpha < \lambda \rangle$. Then \mathbb{P} is also σ -centered.³*

To prove this we would need the following lemma (theorem 3 in [3]):

²The proof of this lemma is an elaboration of an answer on Math Stack Exchange at <http://math.stackexchange.com/q/1766520>

³The proof of this lemma was extracted from the answers of Andreas Blass and Stefan Geschke on Math Overflow at <http://mathoverflow.net/q/84124>

Lemma 3.7. *If $\lambda < (2^{\aleph_0})^+$ then there is a function $F : \lambda \times \omega \rightarrow \omega$ such that for every $\alpha_1, \dots, \alpha_l < \lambda$, $a_1, \dots, a_l < \omega$ there is some $n < \omega$ such that $F(\alpha_i, n) = a_i$ for all $i = 1, \dots, l$.*

Proof of lemma 3.6. For each $\alpha < \lambda$ let $\mathbb{P}_\alpha = \bigcup_{n < \omega} \mathbb{P}_\alpha^n$.

Let $F : \lambda \times \omega \rightarrow \omega$ be a function as in lemma 3.7, and set

$$\mathbb{Q}_n = \left\{ p \in \mathbb{P} \mid \forall \alpha < \lambda \left(p(\alpha) \in \mathbb{P}_\alpha^{F(\alpha, n)} \right) \right\}.$$

First we claim that $\mathbb{P} = \bigcup_{n < \omega} \mathbb{Q}_n$. Let $p \in \mathbb{P}$, and assume $\alpha_1, \dots, \alpha_l < \lambda$ are the only ones where $p(\alpha) \neq \mathbb{1}_{\mathbb{P}_\alpha}$. Let $k_i < \omega$ be such that $p(\alpha_i) \in \mathbb{P}_{\alpha_i}^{k_i}$. Let n be such that $F(\alpha_i, n) = k_i$ for $i = 1, \dots, l$. So $p(\alpha_i) \in \mathbb{P}_{\alpha_i}^{F(\alpha_i, n)}$ for every $i = 1, \dots, l$, and for every other α we have $p(\alpha) = \mathbb{1}_{\mathbb{P}_\alpha} \in \mathbb{P}_\alpha^{F(\alpha, n)}$ (by the assumption in remark 3.2), so $p \in \mathbb{Q}_n$.

Second we claim that each \mathbb{Q}_n is centered. Let $p_1, \dots, p_l \in \mathbb{Q}_n$. For every $\alpha < \lambda$, if there is some i such that $p_i(\alpha) \neq \mathbb{1}_{\mathbb{P}_\alpha}$, we define $q(\alpha)$ to be some common extension of $p_1(\alpha), \dots, p_l(\alpha)$ in \mathbb{P}_α , which exists since $\mathbb{P}_\alpha^{F(\alpha, n)}$ is centered. Otherwise we take $q(\alpha) = \mathbb{1}_{\mathbb{P}_\alpha}$. Only finitely many α 's will give us $q(\alpha) \neq \mathbb{1}_{\mathbb{P}_\alpha}$, so indeed $q \in \mathbb{P}$, and it is clear that it is a common extension of p_1, \dots, p_l . \square

The two-step iteration of σ -centered posets is also σ -centered:⁴

Lemma 3.8. *If \mathbb{P} is a σ -centered posets and $\dot{\mathbb{Q}}$ is a \mathbb{P} -name such that \mathbb{P} forces that $\dot{\mathbb{Q}}$ is a σ -centered posets, then also $\mathbb{P} * \dot{\mathbb{Q}}$ is σ -centered.*

Proof. Let $\mathbb{P} = \bigcup_{n < \omega} \mathbb{P}_n$ such that each \mathbb{P}_n is centered. By remark 3.2 we assume that each \mathbb{P}_n is upward closed. By the assumption, there are \mathbb{P} -names $\dot{\mathbb{Q}}_n$ such that $\mathbb{1}_{\mathbb{P}}$ forces that $\dot{\mathbb{Q}} = \bigcup_{n < \omega} \dot{\mathbb{Q}}_n$ and each $\dot{\mathbb{Q}}_n$ is centered. Let

$$A_{n,m} = \left\{ \langle p, \tau \rangle \in \mathbb{P} * \dot{\mathbb{Q}} \mid \exists q \in \mathbb{P}_n \left(q \leq p \wedge q \Vdash \tau \in \dot{\mathbb{Q}}_m \right) \right\}.$$

- $\mathbb{P} * \dot{\mathbb{Q}} = \bigcup_{n,m < \omega} A_{n,m}$: If $\langle p, \tau \rangle \in \mathbb{P} * \dot{\mathbb{Q}}$, since $p \Vdash \dot{\mathbb{Q}} = \bigcup_{n < \omega} \dot{\mathbb{Q}}_n$ and by the definition of the iteration $p \Vdash \tau \in \dot{\mathbb{Q}}$, there is some $q \leq p$ s.t. $q \Vdash \tau \in \dot{\mathbb{Q}}_m$ for some $m < \omega$. In addition there is some n s.t. $q \in \mathbb{P}_n$, therefore $\langle p, \tau \rangle \in A_{n,m}$.
- Each $A_{n,m}$ is centered: Let $\langle p_i, \tau_i \rangle \in A_{n,m}$ for $i = 1, \dots, k$ and let $q_i \Vdash \tau_i \in \dot{\mathbb{Q}}_m$. Since \mathbb{P}_n is centered we can take a common extension $q \in \mathbb{P}$ of the q_i 's. So $q \Vdash \tau_i \in \dot{\mathbb{Q}}_m$ for each i and also q forces that $\dot{\mathbb{Q}}_m$ is centered, so it forces that there a common extension for them. So there is some name τ such that q forces that τ is a common extension of all the τ_i 's in $\dot{\mathbb{Q}}$. In particular $q \Vdash \tau \in \dot{\mathbb{Q}}$ so $\langle q, \tau \rangle \in \mathbb{P} * \dot{\mathbb{Q}}$ and by definition it extends each $\langle p_i, \tau_i \rangle$.

So indeed $\mathbb{P} * \dot{\mathbb{Q}}$ is a countable union of centered posets as required. \square

Note that this shows that $\Gamma_{\sigma\text{-centered}}$, the class of all σ -centered forcing notions, is transitive. It is also reflexive since the trivial forcing is trivially σ -centered, and persistent since being the union of countably many centered subsets is an upward absolute notion. So, using theorem 2.8, we have the following:

Theorem 3.9. *The ZFC-provable principles of σ -centered forcing contain S4.2.*

⁴Actually something stronger is true - similarly to the previous proof, it is shown by Andreas Blass at <http://mathoverflow.net/q/84129> that the iteration with finite support of less than $(2^{\aleph_0})^+$ σ -centered posets is again σ -centered. We however do not need this fact here.

Finally, it will be of use to know that essentially, σ -centered forcing notions are “small”, so there aren’t too many of them:

Lemma 3.10. *Let \mathbb{P} be a σ -centered forcing notion. Then the separative quotient of \mathbb{P} is of size at most 2^{\aleph_0} .*

Proof. Let $\mathbb{P} = \bigcup_{n \in \omega} \mathbb{P}_n$. Again we assume that each \mathbb{P}_n is upward closed. Recall that the separative quotient is the quotient of \mathbb{P} by the equivalence relation: $x, y \in \mathbb{P}$, $x \sim y$ iff $\{z \in \mathbb{P} \mid z \parallel x\} = \{z \in \mathbb{P} \mid z \parallel y\}$. We denote the equivalence class of x by $[x]$. Define for every $x \in \mathbb{P}$

$$A(x) = \{n \in \omega \mid x \in \mathbb{P}_n\}.$$

We claim that for every $x, y \in \mathbb{P}$, $[x] \neq [y]$ implies $A(x) \neq A(y)$. $[x] \neq [y]$ means that (WLoG) there is some $z \parallel x$ such that $z \perp y$. Let z' be a common extension of z and x . So $z' \perp y$ as well (otherwise z would be compatible with y). Let $n \in \omega$ such that $z' \in \mathbb{P}_n$. Since we assumed \mathbb{P}_n is upward closed, also $x \in \mathbb{P}_n$, so $n \in A(x)$. Assume towards contradiction that $n \in A(y)$, i.e. $y \in \mathbb{P}_n$. But \mathbb{P}_n is centered, so y and z' must be compatible, which is a contradiction.

So we get that $|\mathbb{P}/\sim| \leq |\{A(x) \mid x \in \mathbb{P}\}| \leq |\mathcal{P}(\omega)| = 2^{\aleph_0}$. \square

Corollary 3.11. *Up to forcing-equivalence, there are at most $2^{2^{\aleph_0}}$ σ -centered forcing notions.*

Proof. Every poset is forcing equivalent to its separative quotient (cf. pages 204-206 in [8]), and by the previous lemma there are at most $2^{2^{\aleph_0}}$ of those. \square

3.2. Almost disjoint forcing. In this section we present one of the tools for labeling frames with σ -centered forcing - almost disjoint forcing, which is a version of the example introduced in the previous section. The results in this section are due to Jensen and Solovay in [9].

Two infinite sets are called *almost disjoint* (a.d.) if their intersection is finite. We would like to have a way to construct almost disjoint subsets of ω in a very definable and absolute way. For that, we fix some recursive enumeration $\{s_i \mid i < \omega\}$ of all finite sequences of ω , and define for every $f : \omega \rightarrow \omega$

$$\mathcal{S}(f) = \{i < \omega \mid s_i \text{ is an initial segment of } f\}.$$

If f, g are distinct then $\mathcal{S}(f)$ and $\mathcal{S}(g)$ are almost disjoint. Hence, $\{\mathcal{S}(f) \mid f : \omega \rightarrow \omega\}$ is a family of 2^{\aleph_0} pairwise a.d. subsets of ω .

Recall that for every $Y \subseteq \mathcal{P}(\omega)$, \mathbb{P}_Y is the forcing notion consisting of pairs $\langle s, t \rangle$ where s is a finite subset of ω and t a finite subset of Y , and $\langle s', t' \rangle$ extends $\langle s, t \rangle$ if $s \subseteq s'$, $t \subseteq t'$ and $\forall A \in t, s \cap A = s' \cap A$.

From the discussion at section 2.4 of [9] we obtain the following:

Theorem 3.12. *Let $\mathcal{F} \in M$ be a family of a.d. subsets of ω , and $Y \subseteq \mathcal{F}$ (in M). Then forcing with \mathbb{P}_Y adds a real x such that for every $y \in \mathcal{F}$, $x \cap y$ is finite iff $y \in Y$.*

So \mathbb{P}_Y adds a generic real x which is almost-disjoint from each member of Y . Furthermore, if x is obtained by the generic filter G , then clearly $M[G] = M[x]$. This gives us a method to code subsets of 2^ω using subsets of ω . Let M be some model of ZFC, set as before a recursive enumeration $\{s_i\}$ of $\omega^{<\omega}$ and an enumeration $\{f_\alpha \mid \alpha < \kappa\} \in M$ of ω^ω (where $\kappa = (2^\omega)^M$), and define $\mathcal{F} = \{\mathcal{S}(f_\alpha) \mid \alpha < \kappa\}$. So for each $A \subseteq \kappa$, $A \in M$, we can define $Y = Y(A) = \{\mathcal{S}(f_\alpha) \mid \alpha \in A\}$, and force with \mathbb{P}_Y to obtain a generic

real $x = x_A$. So by the previous theorem, $\alpha \in A$ iff $\mathcal{S}(f_\alpha) \in Y$ iff $x \cap \mathcal{S}(f_\alpha)$ is finite. So, in $M[x]$, we get that

$$A = \{\alpha < \kappa \mid \mathcal{S}(f_\alpha) \cap x \text{ is finite}\}$$

(note that \mathbb{P}_Y preserves both cardinals and the continuum, so $\kappa = (2^\omega)^{M[x]}$, and if $\kappa = \aleph_\alpha^M$ for some α , then also $\kappa = \aleph_\alpha^{M[x]}$). In this case, we say that “ x codes A ”.

4. LABELING FRAMES WITH σ -CENTERED FORCING

Our goal in this section will be to prove that the modal logic of σ -centered forcing is contained in S4.2. Recall that our method for that is labeling finite pre-Boolean-algebras (theorem 2.11). In theorem 2.13 we have shown that a labeling can be produced by finding arbitrarily large families of independent buttons, which are independent of an n -switch for some large enough n . Our proof here will be modeled on that proof, although we will not actually find such families. We begin by describing a specific model W which will be our ground model, then construct an independent family of buttons and two different n -switches. Like in the proof of theorem 2.13, given some finite pBA, we will use the buttons to label the position in the quotient BA, and the n -switches to label the position within each cluster. The difference will be that we use one n -switch to label clusters below the top cluster of the BA, and the other for the top cluster. The need for this modification will be explained in due time.

4.1. The ground model. We begin with the constructible universe L , and use Cohen forcing to obtain mutually generic reals $\langle a_{\alpha,i} \mid \alpha < \omega_1^L, i < \omega \rangle$ i.e. each $a_{\beta,j}$ is generic over $L[\langle a_{\alpha,i} \mid \alpha < \omega_1^L, i < \omega, (\alpha,i) \neq (\beta,j) \rangle]$. Let

$$Z = L[\langle a_{\alpha,i} \mid \alpha < \omega_1^L, i < \omega \rangle].$$

Our ground model W is a generic extension of Z , which preserves the mutual genericity of $\langle a_{\alpha,i} \mid \alpha < \omega_1^L, i < \omega \rangle$, such that these reals are ordinal-definable with a definition which is absolute for generic extensions of W by σ -centered forcing. This can be done e.g. by using Easton forcing to code the reals in the power function above some large enough cardinal. Any extension of W for which the above definition is absolute will be called an *appropriate extension*. We will also require that in W we do not collapse cardinals and add no new subsets below \aleph_ω , so e.g. $\omega_1^L = \omega_1^W$. From now on we'll deal with forcings which do not collapse cardinals (by c.c.c), and also do not change the continuum (by lemma 3.5), so we omit such superscripts.

4.2. The buttons. Now over W we can define T_i , for $i < \omega$, as the statement:

For every real x and for all but boundedly many $\alpha < \omega_1$, $a_{\alpha,i}$ is Cohen generic over $L[x, \langle a_{\beta,j} \mid \beta < \omega_1, j \neq i \rangle]$.

Since the reals $a_{\alpha,i}$ and the sequences $\langle a_{\beta,j} \mid \beta < \omega_1, j \neq i \rangle$ are ordinal definable in W and it's appropriate extensions, also $L[x, \langle a_{\beta,j} \mid \beta < \omega_1, j \neq i \rangle]$ is definable with x as a parameter. So, formally, T_i includes the definitions of these elements, which will be interpreted as we expect in all relevant models. The question whether a real r is generic over some definable submodel is also expressible in the language of set theory, as it just means that r is in every open dense subset of ω^ω which is in that model. So T_i is indeed a sentence in the language of set theory. Note that if we want, by slight abuse of notation we can treat i as a “variable” denoting a natural number, rather than a definable term; thus we would be able to phrase sentences such as $\forall i < \omega T_i$. This will be used in the next section. In this section when we talk about a specific T_i , we take i to be a fixed term.

Remark 4.1. (1) $W \models T_i$ for every i : We required that we do not add any new subsets of ω^ω or any new real. So every real $x \in W$ is already in Z . Fix some $i < \omega$ and a real $x \in W$. This real was introduced by at most boundedly many $a_{\alpha,i}$, that is, there is some $\gamma < \omega_1$ such that

$$x \in L[\langle a_{\alpha,j} \mid \alpha < \omega_1, j \neq i \rangle \cup \langle a_{\alpha,i} \mid \alpha < \gamma \rangle].$$

All the reals $a_{\alpha,i}$ for $\alpha > \gamma$ are generic over the above model so also above $L[x, \langle a_{\beta,j} \mid \beta < \omega_1, j \neq i \rangle]$.

- (2) $\neg T_i$ is a pure button for appropriate extensions: if for some $a_{\alpha,i}$ there is some real x such that $a_{\alpha,i}$ is not generic over $L[x, \langle a_{\beta,j} \mid \beta < \omega_1, j \neq i \rangle]$, then it will never again be generic over this model. So, if we destroy T_i , we can never get it back as long as it keeps its above meaning. Note that if an extension is not appropriate, then T_i might have a completely different meaning than what is intended, as the definitions we use will give some different sets, so it is paramount we stick with appropriate extensions.

We will now define forcing notions which will allow us to destroy T_i , by destroying the genericity of the relevant $a_{\alpha,i}$'s.

Definition 4.2. In W , we define \mathbb{P}_i to be the forcing notion with conditions of the form $\{U_{s_1}, \dots, U_{s_n}, a_{\alpha_1,i}, \dots, a_{\alpha_l,i}\}$ where $n, l < \omega$ and $U_{s_k} \subseteq \omega^\omega$ are basic open (see definition 2.1), and for conditions $p, q \in \mathbb{P}_i$, $q \leq p$ iff $p \subseteq q$ and whenever $a_{\alpha,i} \in p$ and $U_s \in q \setminus p$, $a_{\alpha,i} \notin U_s$. That is, to extend a condition, we can add any finite number of the reals, and we can add any finite number of basic open sets, as long as the new sets do not include any of the old reals.

We will show that the forcing \mathbb{P}_i destroys the genericity of all the $a_{\alpha,i}$'s, by adding dense open sets (approximated by the U_s 's) that do not include them. So, intuitively, a condition $p = \{U_{s_1}, \dots, U_{s_n}, a_{\alpha_1,i}, \dots, a_{\alpha_l,i}\}$ states which reals will be avoided in subsequent stages.

Remark 4.3. Given some distinct $a_{\alpha_1,i}, \dots, a_{\alpha_l,i}$, we can always find some s such that $a_{\alpha_1,i}, \dots, a_{\alpha_l,i} \notin U_s$: let $t = \bigcap_{k=1}^l a_{\alpha_k,i}$, i.e. the longest initial segment common to $a_{\alpha_1,i}, \dots, a_{\alpha_l,i}$. Assume t is of length n , and take some $j \in \omega \setminus \{a_{\alpha_1,i}(n), \dots, a_{\alpha_l,i}(n)\}$. Set $s = t \smallfrown \langle j \rangle$, then $a_{\alpha_1,i}, \dots, a_{\alpha_l,i} \notin U_s$.

Let $G \subseteq \mathbb{P}_i$ be a generic filter. Note that by the former remark, the set of conditions having at least n basic-open sets in them is dense in \mathbb{P}_i (given a condition p , we can find an s such that U_s does not contain any of the reals in p , and then add to p , e.g., $U_s, U_{s \smallfrown \langle 0 \rangle}, U_{s \smallfrown \langle 0,0 \rangle} \dots$ to obtain an extension with at least n basic-open sets). So, the conditions in G give us an infinite sequence $\langle U_{s_n} \mid n < \omega \rangle$ of basic-open sets.

Lemma 4.4. For every $k < \omega$, the set $\bigcup_{n \geq k} U_{s_n}$ is open-dense in ω^ω .

Proof. It is clearly open as a union of open sets. Let $s \in \omega^{<\omega}$, denote by N the maximal length of s_0, \dots, s_{k-1} and let $D_{s,N} = \{p \in \mathbb{P}_i \mid \exists t \supseteq s (\ell(t) > N \wedge U_t \in p)\}$. It is open-dense in \mathbb{P}_i : open is clear, dense since if $p = \{U_{t_1}, \dots, U_{t_n}, a_{\alpha_1,i}, \dots, a_{\alpha_l,i}\}$ is not in $D_{s,N}$, as in remark 4.3 take $t = \bigcap_{k=1}^l a_{\alpha_k,i}$, if $s \not\supseteq t$ then already any extension of s is a legitimate addition to p , so we choose some $s' \supseteq s$ with $\ell(s') > N$ and then $p \cup \{U_{s'}\} \in D_{s,N}$; otherwise we take $s' = t \smallfrown \langle j \rangle$ as in the remark, extend it to some s'' with $\ell(s'') > N$ and $p \cup \{U_{s''}\} \in D_{s,N}$.

So there is some $p \in G \cap D_{s,N}$, and some $U_{s'} \in p$ witnessing that. $p \in G$ so $U_{s'} \in \langle U_{s_n} \mid n < \omega \rangle$, and since $\ell(s') > N$, by the definition of N $s' = s_n$ for some

$n \geq k$, so $U_{s'}$ appears in $\bigcup_{n \geq k} U_{s_n}$. Hence s is extended by some member of $\bigcup_{n \geq k} U_{s_n}$. This was for any s , so $\bigcup_{n \geq k} U_{s_n}$ is indeed dense. \square

Lemma 4.5. *For every $\alpha < \omega_1$ there is some k such that $a_{\alpha,i} \notin \bigcup_{n \geq k} U_{s_n}$.*

Proof. Fix $\alpha < \omega_1$ and let $D_\alpha = \{p \in \mathbb{P}_i \mid a_{\alpha,i} \in p\}$. So D_α is clearly open, and it is dense since for every p , $p \cup \{a_{\alpha,i}\}$ is a legitimate extension of p (we did not limit the addition of $a_{\beta,i}$'s). So there is some $p \in G \cap D_\alpha$. Let k be larger than any n such that $U_{s_n} \in p$. We want to show that $a_{\alpha,i} \notin \bigcup_{n \geq k} U_{s_n}$. Otherwise, there is some $n \geq k$ such that $a_{\alpha,i} \in U_{s_n}$. So there is some $q \in G$ with $U_{s_n} \in q$, and by moving to a common extension (since G is a filter) we can assume $q \leq p$. In fact, $q < p$, since $U_{s_n} \notin p$ by the choice of k and n . But $a_{\alpha,i} \in p$, $q < p$ and $U_{s_n} \in q \setminus p$ imply that $a_{\alpha,i} \notin U_{s_n}$, by contradiction. \square

So indeed, \mathbb{P}_i adds open-dense sets which destroy the genericity of every $a_{\alpha,i}$. This will show that T_i is destroyed, once we show that T_i still means the same thing after forcing with \mathbb{P}_i .

Lemma 4.6. *\mathbb{P}_i is σ -centered.*

Proof. For every $t_1, \dots, t_n \in \omega^{<\omega}$, let $\mathbb{P}(t_1, \dots, t_n)$ be the set of all conditions in \mathbb{P}_i containing exactly the basic-open sets U_{t_1}, \dots, U_{t_n} . Note that there are only ω such sets $\{t_1, \dots, t_n\}$, and that clearly $\mathbb{P}_i = \bigcup_{t_1, \dots, t_n \in \omega^{<\omega}} \mathbb{P}(t_1, \dots, t_n)$. Now notice that every $\mathbb{P}(t_1, \dots, t_n)$ is centered, since if $p_1, \dots, p_l \in \mathbb{P}(t_1, \dots, t_n)$, then $p_1 \cup \dots \cup p_l$ is still a legitimate condition in \mathbb{P}_i , and it extends each p_j since the only limitation on extension concerned the basic-open sets, which we did not change. \square

Corollary 4.7. *Let W' be some appropriate extension of W . Let $G \subseteq \mathbb{P}_i$ be generic over W' . Then $W'[G] \models \neg T_i$.*

Proof. Note that by σ -centeredness, after forcing with \mathbb{P}_i the meaning of all the definitions in T_i remain the same. So we will find a real $x \in W'[G]$ such that all the $a_{\alpha,i}$'s are already not generic over $L[x]$, so surely T_i fails. Let $\langle t_i \mid i < \omega \rangle \in L$ be some definable enumeration of $\omega^{<\omega}$. Define $x = \{m \mid \exists p \in G (U_{t_m} \in p)\}$. So, if as before $\langle U_{s_n} \mid n < \omega \rangle$ is the sequence of basic-open sets given by G , then it is equivalent to $\langle U_{t_m} \mid m \in x \rangle$ by some rearranging of the order. So in $L[x]$ we can already define each union $\bigcup_{n \geq k} U_{s_n}$. Hence, as we have shown above, each $a_{\alpha,i}$ is not in some dense-open set of ω^ω in $L[x]$, and therefore not generic over $L[x]$ as required. \square

Our next task will be to show that forcing with some \mathbb{P}_j does not affect the truth of T_i for any $i \neq j$.

Lemma 4.8. *Let W' be an appropriate extension of W , such that $W' \models T_i$. Let $G \subseteq \mathbb{P}_j$ be generic over W' for $j \neq i$. Then $W'[G] \models T_i$.*

Proof. Assume otherwise. Then there is some $x \in W'[G]$ such that unboundedly many $a_{\alpha,i}$ are not generic over $L[x, \langle a_{\beta,k} \mid \beta < \omega_1, k \neq i \rangle]$. Let $\dot{x} \in W'$ be a \mathbb{P}_j -name for x . Since x is a real, we can assume that \dot{x} is a name containing only elements of the form $\langle q, \tilde{n} \rangle$ for $n \in \omega$ and $q \in \mathbb{P}_j$. Furthermore, since \mathbb{P}_j is c.c.c, we can assume that there are only countably many elements of the form $\langle q, \tilde{n} \rangle$ for each n . So \dot{x} is a countable collection of elements of the form $\langle q, \tilde{n} \rangle$. We wish to “code” \dot{x} by some real $y \in W'$. We do this in the usual way:

- Let γ be the supremum of all $\alpha < \omega_1$ such that $a_{\alpha,j} \in q$ for some $\langle q, \check{n} \rangle \in \dot{x}$. Since \dot{x} is countable and each such q contains only finitely many $a_{\alpha,j}$'s, $\gamma < \omega_1$.
- Each $q \in \mathbb{P}_j$ is of the form $\{U_{s_1}, \dots, U_{s_n}, a_{\alpha_1,j}, \dots, a_{\alpha_l,j}\}$, so it is determined by a finite subset of $\omega^{<\omega}$ and a finite subset of ordinals no larger than γ . This information can be coded by a finite sequence of natural numbers z_q (e.g. a sequence s is coded by $3^{s(0)} \cdot 5^{s(1)} \dots p_{\ell(s)}^{s(\ell(s)-1)}$ where p_m is the m -th prime number, and fixing some bijection $f : \gamma \rightarrow \omega$ we code $a_{\alpha,j}$ using $2^{f(\alpha)+1}$).
- Each pair $\langle z_q, \check{n} \rangle$ can be coded by a natural number (e.g. if $\ell(z_q) = m$ then by $2^n \cdot 3^{z_q(0)} \dots p_{\ell(z_q)}^{z_q(m-1)}$).
- So the entire \dot{x} can be coded by a set of natural numbers y . All these codings are done in W' so $y \in W'$.

Now assume that $W'[G] \models "a_{\alpha,i} \text{ is not generic over } M' := L[x, \langle a_{\beta,k} \mid \beta < \omega_1, k \neq i \rangle]"$. We'll show that already $W' \models "a_{\alpha,i} \text{ is not generic over } M := L[y, \langle a_{\beta,k} \mid \beta < \omega_1, k \neq i \rangle]"$.

$M \subseteq W'$, and since the definition of \mathbb{P}_j requires only the reals $\langle a_{\beta,j} \mid \beta < \omega_1 \rangle$, $\mathbb{P}_j \in M$. In addition, since we can decode y in this model, we have $\dot{x} \in M$. Since $\mathbb{P}_j \in M \subseteq W'$, G is generic also over M . The fact that $a_{\alpha,i}$ is not generic over M' means that there is a dense open set $U \in M'$ such that in $W'[G]$, $a_{\alpha,i} \notin U$. Since $\dot{x} \in M$, $M' \subseteq M[G]$, so $U \in M[G] \subseteq W'[G]$. So there is some $p \in G$ and some \mathbb{P}_j -name $\dot{U} \in M$ such that $p \Vdash "\dot{U} \text{ is an open-dense subset of } \omega^\omega$, and $\check{a}_{\alpha,i} \notin \dot{U}"$. Define

$$\bar{U} = \left\{ r \in \omega^\omega \mid \exists p' \leq p \left(p' \Vdash \check{r} \in \dot{U} \right) \right\}.$$

So $\bar{U} \in M$. We claim first that \bar{U} is open-dense.

Open: let $r \in \bar{U}$, witnessed by $p' \leq p$ s.t. $p' \Vdash \check{r} \in \dot{U}$. Since p' also forces that \dot{U} is open, there is some $p'' \leq p'$ and some $s \in \omega^{<\omega}$ such that $p'' \Vdash "\check{r} \in U_s \subseteq \dot{U}"$, that is $p'' \Vdash \check{s} \sqsubseteq \check{r} \wedge (\check{s} \sqsubseteq \check{r}' \rightarrow \check{r}' \in \dot{U})$. The initial segment relation does not change, so $s \sqsubseteq r$. If $r' \in U_s$, then in particular we'll get $p'' \Vdash \check{r}' \in \dot{U}$, so by definition $r' \in \bar{U}$. So $U_s \subseteq \bar{U}$. So \bar{U} is open.

Dense: Let $s \in \omega^{<\omega}$. Since p forces that \dot{U} is open-dense, there is some $p' \leq p$ and some $t \sqsupseteq s$ such that $p' \Vdash (\check{t} \sqsubseteq \check{r} \rightarrow \check{r} \in \dot{U})$. So let some $t \sqsubseteq r \in \omega^\omega$, then in particular we get $p' \Vdash \check{r} \in \dot{U}$, so $r \in \bar{U}$ and $r \sqsupseteq s$ as required.

Second, we claim that in W' , $a_{\alpha,i} \notin \bar{U}$. Otherwise, there is some $p' \leq p$ such that $p' \Vdash \check{a}_{\alpha,i} \in \dot{U}$. But p was a condition forcing $\check{a}_{\alpha,i} \notin \dot{U}$, a contradiction.

So, we have found an open-dense set $\bar{U} \in M$ such that $a_{\alpha,i} \notin \bar{U}$, so $a_{\alpha,i}$ is not generic over M . This was for every $a_{\alpha,i}$ not generic over $L[x, \langle a_{\beta,k} \mid \beta < \omega_1, k \neq i \rangle]$, and we assumed there are unboundedly many of these. So there are unboundedly many $a_{\alpha,i}$'s which are not generic over $M = L[y, \langle a_{\beta,k} \mid \beta < \omega_1, k \neq i \rangle]$, where $y \in W'$. But this contradicts the assumption that $W' \models T_i$. So, we indeed get that also $W'[G] \models T_i$. \square

To conclude, packing up what we have done in this section, we obtain the following:

Theorem 4.9. $\{\neg T_i \mid i < \omega\}$ is a family of independent buttons over W for σ -centered forcing.

Remark 4.10. In fact, we can replace " σ -centered" with any every reflexive and transitive class of forcing notions, containing all the \mathbb{P}_i 's, such that every extension of W with a forcing from the class yields an appropriate extension.

Note that if it were the case that in no extension of W by σ -centered forcing all these buttons are pushed, we could have finished the proof of our main theorem using corollary 2.16. However, by lemma 3.6, $\prod_{i < \omega} \mathbb{P}_i$ is σ -centered, and it pushes all the buttons. Therefore we need something more to complete the proof.

4.3. The n -switches. In this section we would like to define an n -switch which is independent of any finite number of the buttons from the previous section, using a construction similar to clause 2. of lemma 2.15. We begin with a construction which gives us statements which are almost an n -switch, as they function only for specific extensions of W . Then we build a second n -switch that will function for all other extensions, but will not be independent of the buttons. In the next section we will show how to use both constructions for our purpose.

Given an enumeration of some infinite number of the statements from the previous section, $\langle T_i \mid 0 < i < \omega \rangle$ (we can rename the statements as we want, and later we will use this fact), define R_j (where $j < \omega$) as the statement $j = \sup(\{0 < i < \omega \mid \neg T_i\} \cup \{0\})$ (recall that we can use i as a “variable” in T_i). So R_0 holds iff no button is pushed, and if in some appropriate extension of W we have R_j for $j > 0$, then in particular we have $\neg T_j \wedge T_l$ for any $l > j$, and we can force with \mathbb{P}_l to obtain exactly R_l . Additionally, if some R_j holds, it means in particular that the number of pushed buttons is bounded. Now, to define an n -switch, given some $n > 1$, we set $\Theta_j = \exists k < \omega (R_k \wedge k \bmod n = j)$ for any $j < n$. So in any appropriate extension of W' , if Θ_j holds for some j , this means in particular that there is some k such that R_k holds, so for every $j' < n$ we can find $k' > k$ with $k' \bmod n = j'$ and then force with $\mathbb{P}_{k'}$ to obtain $R_{k'}$ and thus $\Theta_{j'}$. It is also clear that no two distinct Θ_j 's can hold at the same time, and that if the above supremum is finite, then some Θ_j hold. So, $\{\Theta_j \mid j < n\}$ functions as an n -switch, but only as long as the number of pushed buttons is bounded. If in some appropriate extension there are unboundedly many buttons pushed (which is possible as we noted above), no R_k holds, so also no Θ_j hold. So this is “almost” an n -switch.

We now define a “real” n -switch, which we will use in extensions where the number of pushed buttons may be unbounded. The reason we don't use only the following n -switch is that it may not be independent from the buttons.

Proposition 4.11. *Let W be the model constructed in the previous section. Then for every $n > 1$ there is an n -switch for σ -centered forcing over W .*

Proof. Recall that $Z = L[\langle a_{\alpha,i} \mid \alpha < \omega_1, i < \omega \rangle]$ and that in our construction of W we required that all subsets of ω and of ω_1 in W are already in Z (and in particular $W \models 2^{\aleph_0} = \aleph_1$ and $W \models 2^{\aleph_1} = \aleph_2$). Also note that using the coding of $\langle a_{\alpha,i} \mid \alpha < \omega_1, i < \omega \rangle$, in all appropriate extensions of W , sets in Z are definable from ordinals using sentences of set-theory. So let $\langle f_\alpha \mid \alpha < \omega_1 \rangle \in Z$ be a definable enumeration of all the functions $f : \omega \rightarrow \omega$ in Z (equivalently, in W), and define $y_\alpha = S(f_\alpha)$ as in section 3.2. Let $\langle A_\xi \mid \xi < \omega_2 \rangle \in Z$ be some fixed definable enumeration of all the subsets (in Z , or, equivalently, in W) of ω_1 . Let $C(x, \xi)$ be the statement “ $x \subseteq \omega$ and $\alpha \in A_\xi \leftrightarrow x \cap y_\alpha$ is finite”, referred to as “ x is a real coding A_ξ ”. By the discussion at the end of section 3.2, for every ξ there is a σ -centered forcing notion $\mathbb{Q}_\xi \in Z$ such that $\mathbb{Q}_\xi \Vdash \exists x C(x, \xi)$. We would like to define a ratchet by letting $r(\alpha)$ be the statement $\alpha = \sup\{\xi < \omega_2 \mid \exists x C(x, \xi)\}$, and use it to construct an n -switch as in lemma 2.15. By defining so, we can indeed always increase the value of the $r(\alpha)$ by forcing with \mathbb{Q}_α . The problem is that in a certain extension, forcing with \mathbb{Q}_α might also add a real coding A_ξ some $\xi > \alpha$. To fix that, we

will define an unbounded set \mathcal{E} such that adding a code for A_α for some $\alpha \in \mathcal{E}$ doesn't add a code for any larger A_ξ .

We work now within W , and fix some σ -centered poset $Q \in W$. Let $\alpha < \omega_2$. We define by induction $\{\alpha_\zeta \mid \zeta < \omega_1\}$. Set $\alpha_0 = \alpha$. If $\alpha_\zeta < \omega_2$ is defined for $\zeta < \omega_1$, let

$$\alpha_{\zeta+1} = \sup \left\{ \beta < \omega_2 \mid Q \times \prod_{\xi \leq \alpha_\zeta} \mathbb{Q}_\xi \Vdash \neg \exists x C(x, \check{\beta}) \right\} + 1.$$

The above set is not empty since $\mathbb{Q}_{\alpha_\zeta} \Vdash \exists x C(x, \check{\alpha}_\zeta)$. In particular, $\alpha_\zeta < \alpha_{\zeta+1}$.

Claim 4.12. $\alpha_{\zeta+1} < \omega_2$.

Proof. We show that for any σ -centered forcing $P \in W$, there is some $\beta_P < \omega_2$ such that $P \Vdash \text{"sup } \{\beta \mid \exists x C(x, \beta)\} \leq \beta_P\text{"}$, i.e. $P \Vdash (\exists x C(x, \check{\beta}) \rightarrow \check{\beta} \leq \check{\beta}_P)$. First of all, since any σ -centered forcing preserves cardinals and also the continuum function, $P \Vdash 2^{\aleph_1} = \aleph_2 > 2^{\aleph_0}$. In particular, $P \Vdash \text{"sup } \{\beta \mid \exists x C(x, \beta)\} < \omega_2\text{"}$, since there cannot be \aleph_2 reals each coding a different subset of ω_1 . Note that there can only be \aleph_0 many possible values for $\sup \{\beta \mid \exists x C(x, \beta)\}$: otherwise there would be uncountably many conditions forcing different values for $\sup \{\beta \mid \exists x C(x, \beta)\}$, therefore an uncountable antichain, contradicting the c.c.c of P . Hence, since ω_2 is regular, there is some β_P bounding all these possible values, so it is indeed forced by P that $\sup \{\beta \mid \exists x C(x, \beta)\} \leq \beta_P$.

Now, taking $P = Q \times \prod_{\xi \leq \alpha_\zeta} \mathbb{Q}_\xi$ (since $\alpha_\zeta < \omega_2 = (2^{\aleph_0})^+$, this is σ -centered by lemma 3.6), if for some γ , $P \Vdash \neg \exists x C(x, \check{\gamma})$, then there is $p \in P$ such that $p \Vdash \exists x C(x, \check{\gamma})$, but also $p \Vdash \text{"sup } \{\beta \mid \exists x C(x, \beta)\} \leq \beta_P\text{"}$ so $p \Vdash \check{\gamma} \leq \check{\beta}_P < \check{\omega}_2$. Since the ordinals don't change, we indeed get $\gamma \leq \beta_P$. So by the definition, $\alpha_{\zeta+1} \leq \beta_P + 1 < \omega_2$. \square

For $\zeta < \omega_1$ limit, set $\alpha_\zeta = \sup \{\alpha_\xi \mid \xi < \zeta\}$ (ω_2 is regular so also in this case $\alpha_\zeta < \omega_2$), and finally let $\alpha^* = \sup \{\alpha_\zeta \mid \zeta < \omega_1\}$. Again since ω_2 is regular, $\alpha^* < \omega_2$.

Claim 4.13. Let G be generic for $\mathbb{Q} = Q \times \prod_{\xi < \alpha^*} \mathbb{Q}_\xi$ such that $W[G] \models \exists x C(x, \beta)$. Then $\beta < \alpha^*$.

Proof. Let $x \in W[G]$ such that $W[G] \models C(x, \beta)$. So there is a \mathbb{Q} -name τ and some $p_* \in G$ which forces $C(\tau, \check{\beta})$. For every n , let $C_n \subseteq \mathbb{Q}$ be a maximal antichain below p_* of conditions deciding the statement $\check{n} \in \tau$. By the c.c.c each C_n is countable, so also $C = \bigcup_{n \in \omega} C_n$ is countable. Every element of \mathbb{Q} is of the form $\langle q, (p_\gamma)_{\gamma < \alpha^*} \rangle$ where only for finitely many γ 's $p_\gamma \neq \mathbb{1}_{\mathbb{Q}_\gamma}$. So for each $p \in A$, denote this finite set of ordinals by F_p , and let $\gamma^* = \sup \bigcup_{p \in A} F_p$. Each F_p is a set of ordinals less than α^* , so $\gamma^* \leq \alpha^*$. But since C is countable and each F_p is finite, γ^* has at most countable cofinality, while α^* is the limit of an increasing ω_1 sequence $\langle \alpha_\zeta \mid \zeta < \omega_1 \rangle$, so $\gamma^* < \alpha^*$, and furthermore, there is some $\zeta < \omega_1$ such that $\gamma^* \leq \alpha_\zeta$. Let $\mathbb{Q} = Q \times \prod_{\xi \leq \alpha_\zeta} \mathbb{Q}_\xi$. For every $p \in \mathbb{Q}$, if $p = \langle q, (p_\gamma)_{\gamma < \alpha^*} \rangle$, let $\bar{p} = \langle q, (p_\gamma)_{\gamma \leq \alpha_\zeta} \rangle$, and let $\bar{G} = \{\bar{p} \mid p \in G\}$. \bar{G} is $\bar{\mathbb{Q}}$ -generic over W . We claim that $x \in W[\bar{G}]$ and $W[\bar{G}] \models C(x, \beta)$. Define the $\bar{\mathbb{Q}}$ -name

$$\bar{\sigma} = \bigcup_{n \in \omega} (\{\bar{p} \mid p \in C_n, p \Vdash \check{n} \in \tau\} \times \{\check{n}\}).$$

Remark. By the choice of $\alpha_\zeta \geq \gamma^*$, for every $p \in C_n$, if it is of the form $\langle q, (p_\gamma)_{\gamma < \alpha^*} \rangle$, then $p_\gamma = \mathbb{1}$ for every $\gamma > \alpha_\zeta$.

If $n \in x$ then there is some $p \in G, p \leq p_*$ that forces $\check{n} \in \tau$. By the maximality of C_n , it intersects G , which is a filter, so a condition in the intersection must also force $\check{n} \in \tau$ (and not $\check{n} \notin \tau$). So we can choose such $p \in C_n \cap G$, and by definition $\langle \bar{p}, \check{n} \rangle \in \bar{\sigma}$. In addition, $\bar{p} \in \bar{G}$, so $n \in \bar{\sigma}_{\bar{G}}$.

If $n \in \bar{\sigma}_{\bar{G}}$ there is some $p \in C_n, p \Vdash \check{n} \in \tau$ such that $\bar{p} \in \bar{G}$. So, there is some $r \in G$ such that $\bar{r} = \bar{p}$. Note that by the remark r and p are equal in every coordinate where p is not trivial, so $r \leq p$. Therefore also $r \Vdash \check{n} \in \tau$, and $r \in G$ so $n \in x$.

So, we get that $\bar{\sigma}_{\bar{G}} = x$, so $x \in W[\bar{G}]$. $W[\bar{G}] \subseteq W[G]$, and in $W[G]$ we have $C(x, \beta)$, i.e.

$$A_\beta = \{\alpha \mid x \cap y_\alpha \text{ is finite}\},$$

so we can already have this equation in $W[\bar{G}]$ (since the y_α 's don't change), so $W[\bar{G}] \models C(x, \beta)$. Therefore, we have that $Q \times \prod_{\xi \leq \alpha_\zeta} \mathbb{Q}_\xi \Vdash \neg \exists x C(x, \beta)$, so by the definition of $\alpha_{\zeta+1}, \beta \leq \alpha_{\zeta+1} < \alpha^*$, as required. \square

So we have defined an operation $\alpha \mapsto \alpha^*$ for every $\alpha < \omega_2$ (note that this operation was relative to Q). Since $\forall \alpha < \omega_2$ we have $\alpha < \alpha^* < \omega_2$, the set $\{\alpha^* \mid \alpha < \omega_2\}$ is unbounded in ω_2 , so the set \mathcal{C}_Q consisting of all limit points of this set is a club.

By corollary 3.11, in W there are at most $(2^{2^{\aleph_0}})^W$ σ -centered forcing notions up to equivalence, or to be exact, at most $(2^{2^{\aleph_0}})^W$ separative forcing notions. As we noted before, this cardinal is \aleph_2 . Note that $\left\{ \prod_{\alpha < \xi} \mathbb{Q}_\xi \mid \xi \in \mathcal{C}_Q \right\}$ where Q is the trivial forcing are non-equivalent σ -centered posets, so we get exactly \aleph_2 posets. Since every separative σ -centered poset has size at most \aleph_1 , each can be coded as binary relations on ω_1 , so we can assume all such forcings in W are already in Z . Let $\langle Q_\zeta \mid \zeta < \omega_2 \rangle \in Z$ be some definable enumeration of all the separative σ -centered forcing notions in W , and define \mathcal{C} as be the diagonal intersection of $\langle \mathcal{C}_{Q_\zeta} \mid \zeta < \omega_2 \rangle$:

$$\mathcal{C} := \bigtriangleup_{\zeta < \omega_2} \mathcal{C}_{Q_\zeta} = \left\{ \alpha < \omega_2 \mid \alpha \in \bigcap_{\zeta < \alpha} \mathcal{C}_{Q_\zeta} \right\}.$$

\mathcal{C} is also a club in ω_2 (cf. lemma 8.4 in [8]). Now we let

$$\mathcal{E} = \mathcal{C} \cap \{\alpha < \omega_2 \mid \text{cf}(\alpha) = \omega_1\}.$$

\mathcal{E} is unbounded: it is known that the set $\{\alpha < \omega_2 \mid \text{cf}(\alpha) = \omega_1\}$ is stationary in ω_2 (cf. pg. 94 in [8]), so it intersects the club $\{\alpha \in \mathcal{C} \mid \alpha > \gamma\}$ for each $\gamma < \omega_2$. Let $\langle e_\alpha \mid \alpha < \omega_2 \rangle \in Z$ be a (definable) ascending enumeration of \mathcal{E} .

Remark. We would have preferred to work with \mathcal{C} rather than \mathcal{E} . The problem is that the $*$ operation may not be continuous at limits of countable cofinality - to prove continuity, we would like to imitate the proof of claim 4.13, but it requires that the limit is of uncountable cofinality. If the length of the product is of countable cofinality, there might be a real that is not introduced in any bounded product.

Now we can define $r(\alpha)$ as the statement " $\alpha = \min \{\beta < \omega_2 \mid \neg \exists x C(x, e_\beta)\}$ ". These are indeed statements, since the e_β 's are definable over $Z = L[\langle a_{\alpha,i} \mid \alpha < \omega_1, i < \omega \rangle]$, which is definable in W using the decoding of the generic reals. Again note that this definition retains it's intended meaning in every appropriate extension of W . Now given $n > 1$, define Φ_j (for $j < n$) as the statement " $r(\omega \cdot \alpha + k) \rightarrow (k \bmod n = j)$ ". We claim that $\{\Phi_j \mid j < n\}$ is an n -switch for σ -centered forcing over W .

Let $Q \in W$ be some σ -centered poset, and $G \subseteq Q$ generic over W . By lemma 3.5 $W[G]$ satisfies $2^{\aleph_0} = \aleph_1$, so in particular the set $\{\beta < \omega_2 \mid \exists x C(x, e_\beta)\}^{W[G]}$ is bounded, since there are only ω_1 reals, and therefore there cannot be unboundedly many subsets of ω_1 coded by them. So there is some unique $\gamma < \omega_2$ such that $W[G] \models r(\gamma)$. There are some unique $j, k < n$ such that $\gamma = \omega \cdot \alpha + k$ and $k \bmod n = j$, so $W[G] \models \Phi_j$. Hence every σ -centered extension of W satisfies exactly one Φ_j . Now we need to show that for every $j' \neq j$ there is some σ -centered extension of $W[G]$ satisfying $\Phi_{j'}$. Recall the club \mathcal{C}_Q from the above construction. $Q = Q_\xi$ for some $\xi < \omega_2$. By the unboundedness of \mathcal{E} , we can find some γ' such that $e_{\gamma'} > \xi$ and $\gamma' = \omega \cdot \alpha + j'$ for some α . We want to show that we have a generic extension of $W[G]$ satisfying $r(\gamma')$.

Let $H \subseteq \prod_{\zeta < e_{\gamma'}} \mathbb{Q}_\zeta$ be generic over $W[G]$. By the product lemma, $W[G \times H] = W[G][H]$ and $G \times H$ is $Q \times \prod_{\zeta < e_{\gamma'}} \mathbb{Q}_\zeta$ generic over W . For every $\beta < e_{\gamma'}$, $\prod_{\zeta < e_{\gamma'}} \mathbb{Q}_\zeta \Vdash \exists x C(x, \beta)$, so $W[G][H] \models \exists x C(x, \beta)$. Recall that $e_{\gamma'}$ is in the diagonal intersection of the clubs \mathcal{C}_{Q_ζ} , so by definition, and since $e_{\gamma'} > \xi$, $e_{\gamma'} \in \bigcap_{\zeta < e_{\gamma'}} \mathcal{C}_{Q_\zeta} \subseteq \mathcal{C}_{Q_\xi}$. So, by the definition of \mathcal{C}_{Q_ξ} , $e_{\gamma'}$ is either of the form δ^* for some δ and the $*$ operation corresponding to Q_ξ , or a limit point of such points. In the first case, we can just apply claim 4.13. In the second case, since $e_{\gamma'} \in \mathcal{E}$ is of uncountable cofinality, we can repeat the proof of claim 4.13 with a sequence $\langle \delta_\zeta^* \mid \zeta < \omega_1 \rangle$ that witnesses $e_{\gamma'} \in \mathcal{C}_{Q_\xi}$, and get that the statement in claim 4.13 is true as well. That is, in both cases, we get that if $W[G \times H] \models \exists x C(x, \beta)$ then $\beta < e_{\gamma'}$. So $e_{\gamma'}$ is the first β such that $W[G \times H] \models \neg \exists x C(x, \beta)$. Since the enumeration of \mathcal{E} is increasing, we get that $\gamma' = \min\{\beta < \omega_2 \mid \neg \exists x C(x, e_\beta)\}$. So $W[G \times H] \models r(\gamma')$, and since $\gamma' = \omega \cdot \alpha + j'$, $W[G \times H] \models \Phi_{j'}$ as required. \square

The forcing notions used in this n -switch add real numbers in a rather uncontrollable way, so it is indeed likely that they might add some real which destroys the genericity of the $a_{\alpha, i}$'s, therefore it is unlikely that this n -switch is independent of the buttons $\neg T_i$. However, by using both constructions presented in this section, we can overcome the drawbacks each of them has.

4.4. The labeling.

Theorem 4.14. *There is a σ -centered labeling for every finite pre-Boolean-algebra.*

Proof. Let $\langle F, \leq \rangle$ be a pBA. WLoG, by adding dummy worlds, we can assume each cluster in F contains n elements for some $1 < n < \omega$ ⁵. Let B be a finite set such that $\langle F/\equiv, \leq \rangle$ is isomorphic to $\langle \mathcal{P}(B), \subseteq \rangle$. So every world in F is of the form w_j^C for some $j < n$ and $C \subseteq B$, and we have $w_j^C \leq w_{j'}^{C'}$ iff $C \subseteq C'$.

Let W be the model constructed in section 4.1. We can assume $B \in W$. Consider the predicates T_i constructed in section 4.2, and rename them by choosing some T_b for every $b \in B$, and renaming the rest as $\langle T_i \mid 0 < i < \omega \rangle$. We will refer to these as the T -predicates. For each $b \in B$ and $0 < i < \omega$ let \mathbb{P}_b and \mathbb{P}_i be the corresponding forcing notions destroying T_b and T_i respectively. Let $\{\Theta_j \mid j < n\}$ be statements constructed as in the beginning of section 4.3 from $\langle T_i \mid 0 < i < \omega \rangle$, and $\{\Phi_j \mid j < n\}$ be the n -switch constructed in proposition 4.11.

⁵If every cluster has only one element then this proof is a bit redundant, as we don't need the n -switches, and we can label the BA only with the buttons from section 4.2.

For every $C \subseteq B$, define

$$\Psi_C = \bigwedge_{b \in C} \neg T_b \wedge \bigwedge_{b \notin C} T_b$$

which states that the pushed buttons out of $\{\neg T_b \mid b \in B\}$ are exactly the ones labeled by elements in C .

Now for every $C \subsetneq B$ and $j < n$ we set $\Phi(w_j^C) = \Psi_C \wedge \Theta_j$. For $C = B$ and $j < n$, we set

$$\Phi(w_j^B) = (\Psi_B \wedge \Theta_j) \vee [(\sup\{n \mid \neg T_n\} = \omega) \wedge \Phi_j].$$

So, each cluster is labeled using the independent buttons $\{\neg T_b \mid b \in B\}$, and to move within each cluster below the topmost one we use the “almost” n -switch $\{\Theta_j \mid j < n\}$. If we can no longer use it, that is, if there are unboundedly many T_i ’s destroyed, we put ourselves in the top cluster, and there we move using the n -switch $\{\Phi_j \mid j < n\}$. In this way, the fact that this n -switch is not independent of the buttons will not affect us, as we will always stay in the top cluster anyway. We will now show that this is indeed a labeling as required.

The statements are mutually exclusive: It is clear that the statements $\{\Psi_C \mid C \subseteq B\}$ are mutually exclusive, so $\Phi(w_j^C), \Phi(w_{j'}^{C'})$ for $C \neq C'$, both different than B , clearly exclude each other. If we look at $\Phi(w_j^C)$ and $\Phi(w_{j'}^B)$ for some $C \neq B$, they exclude each other since if $\Phi(w_{j'}^B)$ holds, then either Ψ_B holds which excludes Ψ_C , or $\sup\{n \mid \neg T_n\} = \omega$ holds, which excludes Θ_j . Now for $j \neq j'$ if $C \subsetneq B$, $\Phi(w_j^C), \Phi(w_{j'}^{C'})$ exclude each other since Θ_j and $\Theta_{j'}$ exclude each other; and if $C = B$, if $\sup\{n \mid \neg T_n\} = \omega$ holds then Φ_j and $\Phi_{j'}$ exclude each other, and otherwise again Θ_j and $\Theta_{j'}$ exclude each other.

The statements exhaust the truth over σ -centered generic extensions of W : Let $W[G]$ be some σ -centered generic extension of W . It is appropriate. If $W[G] \models \sup\{n \mid \neg T_n\} = \omega$, then there is some j such that $W[G] \models \Phi_j$, and so $W[G] \models \Phi(w_j^B)$. Otherwise, the number of buttons pushed is finite, so there is some j such that $W[G] \models \Theta_j$, and there is also some specific subset of the buttons $\{\neg T_b \mid b \in B\}$ which are pushed in $W[G]$, so there is some $C \subseteq B$ such that $W[G] \models \Psi_C$, and together we get $W[G] \models \Phi(w_j^C)$.

W satisfies $\Phi(w_0^\emptyset)$: In W we have T_b for all $b \in B$, so $W \models \Psi_\emptyset$. Also all the T_i ’s hold, so we have R_0 , and therefore also $W \models \Phi_0$.

The statements correspond to the relation: Assume we are in U which is a σ -centered forcing extension of W where $\Phi(w_j^C)$ is true.

Assume first that $C \neq B$.

- Assume $\Diamond \Phi(w_{j'}^{C'})$. That means there is a σ -centered generic extension U' of U satisfying $\Phi(w_{j'}^{C'})$. If $C' \neq B$ then

$$U' \models \Psi_{C'} = \bigwedge_{b \in C'} \neg T_b \wedge \bigwedge_{b \notin C'} T_b.$$

But $U \models \bigwedge_{b \in C} \neg T_b$, which are buttons, so they remain pushed in U' , i.e. $U' \models \bigwedge_{b \in C} \neg T_b$. So we must get $C \subseteq C'$, so $w_j^C \leq w_{j'}^{C'}$. If $C' = B$ then clearly we have $w_j^C \leq w_{j'}^{C'}$.

- Assume $w_j^C \leq w_{j'}^{C'}$, hence $C \subseteq C'$. We have

$$U \models \Psi_C = \bigwedge_{b \in C} \neg T_b \wedge \bigwedge_{b \notin C} T_b,$$

so for every $b \in C' \setminus C$, we can force $\neg T_b$, to obtain an extension U' satisfying $\Psi_{C'}$ (the buttons from C will remain pushed). In U , which satisfies Θ_j , there is some k such that $k \bmod n = j$ and $U \models R_k$. In U' we still have R_k , since pushing buttons of the form $\neg T_b$ does not push any button $\neg T_i$. If $j' = j$ we are done, otherwise we can find some $k' > k$, such that $k' \bmod n = j'$, push $\neg T_{k'}$ and thus obtain an extension U'' satisfying $\Theta_{j'}$. Again this forcing does not affect the T_b 's, so U'' also satisfies $\Psi_{C'}$, so it satisfies $\Phi(w_{j'}^{C'})$. By transitivity, we get that indeed $U \models \Diamond \Phi(w_{j'}^{C'})$.

Now assume $C = B$, i.e. $U \models \Phi(w_j^B)$. We distinguish the two cases.

- First, we assume $U \models (\Psi_B \wedge \Theta_j)$
 - Assume $\Diamond \Phi(w_{j'}^{C'})$. Since $U \models \Psi_B$, any extension of it also satisfies Ψ_B , so we cannot have $\Diamond \Phi(w_{j'}^{C'})$ for any $C' \neq B$. Therefore $C' = B$ and indeed $w_j^B \leq w_{j'}^B = w_{j'}^{C'}$.
 - Assume $w_j^C \leq w_{j'}^{C'}$. So $C' = B$ as well. $U \models \Theta_j$, so as we have seen before, we can force over U to obtain a generic extension satisfying $\Theta_{j'}$. This extension will still satisfy Ψ_B since these are buttons, so it will satisfy $\Phi(w_{j'}^B)$ as required.
- Second assume $U \models (\sup \{n \mid \neg T_n\} = \omega) \wedge \Phi_j$.
 - Assume $\Diamond \Phi(w_{j'}^{C'})$. Since $U \models \sup \{n \mid \neg T_n\} = \omega$, any extension of it also satisfies $\sup \{n \mid \neg T_n\} = \omega$ since these are buttons, so we cannot have $\Diamond \Phi(w_{j'}^{C'})$ for any $C' \neq B$. Therefore $C' = B$ and indeed $w_j^B \leq w_{j'}^{C'}$.
 - Assume $w_j^C \leq w_{j'}^{C'}$. So $C' = B$ as well. $U \models \Phi_j$, so as we have seen before, we can force over U to obtain a generic extension satisfying $\Phi_{j'}$. This extension will still satisfy $\sup \{n \mid \neg T_n\} = \omega$ since these are buttons, so it will satisfy $\Phi(w_{j'}^B)$ as required.

Hence we have defined a σ -centered labeling for the frame $\langle F, \leq \rangle$ over W . □

Corollary 4.15. *If ZFC is consistent, then the ZFC-provable principles of σ -centered forcing are exactly S4.2.*

Proof. If ZFC is consistent then we can obtain the model W and the labelings described above. So by theorem 2.11 the ZFC-provable principles of σ -centered forcing are contained in S4.2, and by theorem 3.9 we get equality. □

5. GENERALIZATIONS AND OPEN QUESTIONS

Throughout this work, we have focused on σ -centered forcing notions. However, by examining the proofs, one can see that we have not used the full strength of σ -centeredness. To obtain the lower bound, we used the reflexivity, transitivity and persistence of σ -centered posets. And to obtain the upper bound, we defined labelings using two main ingredients - the first is the posets constructed section 4.2, giving us the buttons and an “almost” n -switch, and the second is the n -switch of proposition 4.11. To work

with the buttons, we also required that all extensions of W will be appropriate. Assuming this, once we had an n -switch, we did not use its specific construction in defining the labeling. So in fact we have the following:

Theorem 5.1. *Let W be the model constructed in section 4.1 and Γ a class of forcing notions with the following properties:*

- (1) Γ is reflexive, transitive and persistent;
- (2) Every extension of W by a Γ -forcing is appropriate;
- (3) All posets constructed in section 4.2 are in Γ ;
- (4) There is an n -switch for Γ -forcing over W for any n .

Then $\text{MLF}(\Gamma) = \text{S4.2}$.

Now let's see what was needed to obtain the n -switch of proposition 4.11. We relied heavily on the c.c.c of all posets in question; we used all posets coding subsets of ω_1 , as well as products of at most \aleph_1 of them; we relied on the fact that σ -centered posets cannot enlarge 2^{\aleph_0} or 2^{\aleph_1} ; we used the fact that there were (in W) only \aleph_2 σ -centered posets up to equivalence, and that they were all already in the smaller model Z . So, this construction can be carried with any class of forcing notions satisfying these requirements. To conclude:

Theorem 5.2. *Let Γ be a class of forcing notions with the following properties:*

- (1) Γ is reflexive, transitive and persistent;
- (2) Every extension of W by a Γ -forcing is appropriate;
- (3) All posets constructed in section 4.2 are in Γ ;
- (4.1) Each poset in Γ has the c.c.c, and does not enlarge 2^{\aleph_0} or 2^{\aleph_1} ;
- (4.2) $|\Gamma| \leq 2^{2^{\aleph_0}}$ (where the size is measured up to equivalence of forcing);
- (4.3) $\Gamma^W \subseteq Z$;
- (4.4) All posets which are used to code subsets of ω_1 , and products of at most \aleph_1 of them, are in Γ .

Then $\text{MLF}(\Gamma) = \text{S4.2}$.

Remark. Conditions 3 and 4.4 will hold for any class containing all σ -centered forcing notions.

Definition 5.3. A subset $C \subseteq \mathbb{P}$ is called n -linked if any n elements of C are compatible, i.e. have a common extension (perhaps not in C itself). 2-linked is also called simply linked. A poset is called σ - n -linked if it is the union of ω many n -linked subsets. Again, σ -linked means σ -2-linked.

It is clear that we have the following implications:

$$\sigma\text{-centered} \rightarrow \sigma\text{-}n\text{-linked for every } n \rightarrow \sigma\text{-}n\text{-linked} \rightarrow \sigma\text{-linked}$$

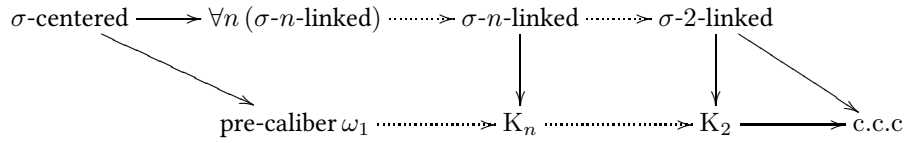
and it is known that the other directions do not hold (cf. [1]). So the classes of all forcing notions satisfying one of these properties contain the class of σ -centered posets. We observe that in the proofs of lemmas 3.5 and 3.10, we actually only used the fact that the posets were σ -linked rather than strictly σ -centered. So each of these classes preserve the continuum function, each such poset is equivalent to a poset of size at most 2^{\aleph_0} , and the class itself has size at most $2^{2^{\aleph_0}}$ (up to equivalence). Additionally, one can see that the proofs of lemmas 3.6 (the strong productivity of the class) and 3.8 (the transitivity of the class) can be applied, *mutatis mutandis*, to the classes of σ - n -linked posets for some n or the class of posets which are σ - n -linked for every n . These classes are also clearly persistent, so they satisfy all the conditions of theorem 5.2, thus we obtain the following:

Corollary 5.4. *Let Γ be either the class of all σ - n -linked posets (for some fixed n), or the class of all posets which are σ - n -linked for every n . Then $\text{MLF}(\Gamma) = \text{S4.2}$.*

Parallel to this hierarchy of properties, we can define the following hierarchy (cf. [1]):

Definition 5.5. (1) Given $n \in \omega$, \mathbb{P} has property K_n if every $A \subseteq [\mathbb{P}]^{\aleph_1}$ contains an uncountable n -linked subset. K_2 is also called the *Knaster property*.
 (2) \mathbb{P} has *pre-caliber* ω_1 if every $A \subseteq [\mathbb{P}]^{\aleph_1}$ contains an uncountable centered subset.

Note that pre-caliber ω_1 implies property K_n , and K_n implies K_m for $m \leq n$. So these form a hierarchy of properties. Furthermore, if \mathbb{P} is σ -centered then it has pre-caliber ω_1 , and if it is σ - n -linked then it has property K_n . So we get the following implications:



Let the property P be either pre-caliber ω_1 or K_n for some n , and denote by Γ_P the class of all P -forcing notions. Then one can verify that Γ_P is reflexive, transitive and persistent (hence directed), so $\text{MLF}(\Gamma_P) \supseteq \text{S4.2}$. Furthermore, it contains all σ -centered forcings, so it would be natural to try and generalize our results to this parallel hierarchy. However, note that for every I , the poset $\text{Fn}(I, 2)$ consisting of finite functions from I to 2, ordered by reverse inclusion, has pre-caliber ω_1 (cf. [11, pg. 181]), and for $|I| > 2^{\aleph_0}$ it adds $|I|$ new reals, so it does not preserve the continuum function. So extensions of W using such forcings may not be appropriate. We can however get some limited result. Note that the coding of the reals $a_{\alpha, i}$ can be started as high as we want, so if we limit ourselves to forcing notions of a bounded size, we can do this coding somewhere high enough that will not be affected by these forcings. So we can proceed with this the labeling, if we show that there is some n -switch to label the top cluster with. Since these forcings do not preserve the continuum, we cannot obtain an n -switch for these classes using the same methods. However, there is a very natural n -switch for any class of c.c.c forcings which contain all forcings of the form $\text{Fn}(I, 2)$: the statements $2^{\aleph_0} \geq \aleph_\alpha$ form a ratchet, since in any extension of the ground model, using an appropriate I , we can always increase the size of the continuum, and by c.c.c it will not drop. If we bound the size of the forcings appropriately, this would be a strong ratchet, and we can construct an n -switch as in lemma 2.15. So all the conditions of theorem 5.1 can be met, hence we have the following:

Corollary 5.6. *Let P be either pre-caliber ω_1 or K_n for some n and Γ_P the class of all P -forcing notions of some bounded size. Then $\text{MLF}(\Gamma_P) = \text{S4.2}$.*

However, to deal with all P -forcings at the same time would require a different method, so the following is open:

Question 5.7. *Let P be either pre-caliber ω_1 or K_n for some n . What is the modal logic of all P -forcing notions?*

To conclude, the only property in the above diagram we did not discuss yet is c.c.c.

Question 5.8. *What is the modal logic of all c.c.c forcing notions?*

This natural question was already raised in [6]. The difficulty in answering it is that the class of all c.c.c forcing notions is *not* directed, so it does not contain S4.2. It is

reflexive and transitive, so Hamkins and Löwe conjectured that the answer is S_4 . To prove this, one would probably need to find a labeling for models based on trees, as the class of all trees is a class of simple frames characterizing S_4 . It should be mentioned that in [7], a labeling of models based frames which are “spiked pre-Boolean algebras” (cf. [7] for exact definition) was provided, thus establishing an upper bound which is strictly between S_4 and $S_{4.2}$. However it is not known whether this modal theory is finitely axiomatizable, so it is not yet clear whether this can be shown to be a lower bound as well by the current methods. So, this question remains open.

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